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LETTER TO THE EDITOR

Non-self-dual monopole solutions for SU(n) Yang–Mills–Higgs systems

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Abstract. Non-self-dual monopole solutions are presented for a class of SU(n) Yang-Mills-Higgs models in three dimensions. The models in question are obtained by dimensional reduction of pure Yang-Mills theories in even $d>4$ dimensions. The Higgs field can belong to the adjoint representation or the adjoint $\oplus$ scalar representation of SU(n). In the former case, constraints on the solutions reduce the number of independent ansatz functions in the spherical symmetry ansatz.

Monopole solutions of the Yang-Mills-Higgs (YMH) system in three dimensions have been studied for some time and many such solutions are known [1-3]. These solutions fall into two distinct categories which we label I and II. YMH field configurations of type I correspond to a Lagrangian density which incorporates a Higgs (self-interaction) potential. This potential, of course, ensures that the Higgs fields have the appropriate non-zero asymptotic limits. The original solution [1] for the SU(2) YMH system belongs to this class. The SU(3) solution [4], with Higgs vacuum proportional to the SU(3) generator λ_8, also belongs to this class; however, this solution is equivalent to the SU(2) solution. It is important to note that no explicit solutions of type I have been discovered yet. On the other hand, explicit solutions of type II are known [2, 3]. YMH field configurations of type II occur in the Bogomolny'i-Prasad-Sommerfield [2, 4, 5] limit, where the coupling constant giving the strength of the Higgs potential vanishes. The solutions of type II are the self-dual monopoles, while those of type I are non-self-dual. It is because the first-order self-duality equations are simple to solve (relative to the equations of motion) that explicit solutions have been found for II but not for I. In all of the above cases the Higgs field lies in the adjoint representation of the gauge group.

In this letter we are concerned with spherically symmetric monopole solutions of type I for the gauge group SU(n). The solutions which we find—one for each value of n—fall into two distinct sets, according to whether n is even or odd. For even n the solutions can be viewed as generalisations of the original SU(2) solution [1]—indeed the $n=2$ solution we find is precisely that solution. For odd n the solutions can be viewed as generalisations of the SU(3) solution [4], and again our $n=3$ solution is just that previously constructed solution. The solutions which we present for $n>3$ are new.

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As our solutions are type I we do not present explicit solutions. For each n the solution is determined by specifying, for the Higgs fields, a particular set of non-zero asymptotic boundary conditions. The choice of asymptotic boundary conditions for the Higgs fields—the Higgs vacuum—is made to facilitate the proof of existence of such finite-energy solutions [6].

The SU(n) YMH model which we use to construct this class of solutions is obtained from a pure Yang-Mills (YM) system in an even d-dimensional space, where $d>4$, by dimensional reduction down to three dimensions. The calculus of dimensional reduction used is based on a particular formulation [7] giving a reduction of two and three dimensions, generalised [8] to reduction of an arbitrary odd number of dimensions. This procedure leads to a Higgs potential

$$\mathrm{Tr}[\eta^2 1+(\Phi+\Theta 1)^2]^2 \tag{1}$$

where Φ is the adjoint Higgs field and Θ and η are constants whose significance will be discussed later. It is just this Higgs potential which will give rise to the correct asymptotic behaviour of the Higgs fields to ensure that finite-energy solutions do occur.

It was previously shown in [8] that subjecting the Chern-Pontryagin integral in even d dimensions to dimensional reduction induced by $M_d=E_3\times S^{d-3}$ yields a surface integral on E_3. Moreover, depending on the asymptotic properties of the integrand (consistent with the asymptotic properties of the original integrand) this integral on E_3 might be non-zero and finite, say equal to μ, the magnetic charge of the field configuration. Subjecting a YM system in d dimensions to the same dimensional reduction results in a YMH system with a Higgs potential in which the only arbitrary parameter is the (radius)$^{-1}$ of the sphere of compactification, η. This residual YMH system may have finite energy for field configurations which also yield a finite magnetic charge μ, though it is clear that the energy in such cases cannot be equal to μ. The presence of a Higgs potential indicates that the field configuration in question is not self-dual. At the same time, however, it is by no means certain that the energy of the residual YMH system can be finite, since it is known [9] that the YM system in higher than four dimensions cannot have finite energy. For the dimensional reduction procedure we consider, however, that the residual YMH system does possess finite-energy field configurations which we now present.

The YMH system in three dimensions under examination is described by the Lagrangian density

$$L=\mathrm{Tr}[\tfrac{1}{2}F_{ij}^2+\tfrac{1}{4}(D_i\phi)^2-(\eta^2 1+\phi^2)^2] \tag{2}$$

where the indices $i, j=1, 2, 3$ and where

$$F_{ij}=\partial_i A_j-\partial_j A_i+[A_i, A_j] \tag{3a}$$

$$D_i\phi=\partial_i\phi+[A_i, \phi] \tag{3b}$$

$$\phi=\Phi+\Theta 1 \tag{3c}$$

and η is the (radius)$^{-1}$ of the sphere of compactification. It is clear that $(1/n)\Theta$ is the trace of ϕ. In principle, Θ could be an independent field, but as that leads to Higgs fields which are not purely adjoint we do not pursue that possibility initially. Consequently we impose the constraint

$$\Theta=\text{constant} \tag{4}$$

in a consistent fashion, i.e. we ensure that (4) is a consistent solution of the equations of motion resulting from (2). It is also to be noted that we are using anti-Hermitian fields, so each term in (2) is negative definite.

We seek spherically symmetric field configurations for the system described by equations (2)-(4). Spherical symmetry in three dimensions has been exhaustively studied [10, 11]. Different types of spherical symmetry can be imposed for the gauge group SU(n) corresponding to the different mappings of SO(3) into SU(n). The embedding for which the third generator of SO(3) takes the form

$$T_3 = -\mathrm{i}\,\mathrm{diag}(\tfrac{1}{2}(n-1), \tfrac{1}{2}(n-1)-1, \ldots, 1-\tfrac{1}{2}(n-1), -\tfrac{1}{2}(n-1)) \tag{5}$$

is the commonly used one, as in this case the spherically symmetric ansatz yields a $(\mathrm{U}(1))^{n-1}$ model [11]. In the case of all other embeddings the spherical symmetry ansatz yields non-Abelian models; these cases will be discussed elsewhere. We adopt the notation of [11], which makes the above choice of embedding, for our ansatz:

$$\boldsymbol{B} = (1/r)M_1'\boldsymbol{i} + (1/r)M_2'\boldsymbol{j} + (1/r^2)([M_1, M_2] - T_3)\boldsymbol{k} \tag{6a}$$

$$\boldsymbol{D}\phi = (1/r)[M_2, \phi]\boldsymbol{i} - (1/r)[M_1, \phi]\boldsymbol{j} + \phi'\boldsymbol{k} \tag{6b}$$

$$(M_1)_{\alpha\beta} = -\tfrac{1}{2}\mathrm{i}(\delta_{\alpha,\beta-1}a_\alpha + \delta_{\alpha-1,\beta}a_\beta) \qquad \text{(no sum)} \tag{6c}$$

$$(M_2)_{\alpha\beta} = -\tfrac{1}{2}\mathrm{i}(\delta_{\alpha,\beta-1}a_\alpha - \delta_{\alpha-1,\beta}a_\beta) \qquad \text{(no sum)} \tag{6d}$$

$$\Phi = -\tfrac{1}{2}\mathrm{i}\,\mathrm{diag}(\phi_1, \phi_2 - \phi_1, \ldots, -\phi_{n-1}) \tag{6e}$$

$$\Theta = -\mathrm{i}\theta. \tag{6f}$$

In these equations the functions ϕ_α, a_α for $\alpha = 1, \ldots, n-1$ and their derivatives with respect to r, namely ϕ'_α, a'_α for $\alpha = 1, \ldots, n-1$, depend only on the radial variable r. θ is a constant whose value will be specified later. In terms of the ansatz functions ϕ_α, a_α, $\alpha = 1, \ldots, n-1$, the energy integral takes the form

$$\begin{aligned} E = \int_0^\infty \mathrm{d}r\big(&4[(a_1')^2 + \ldots + (a_{n-1}')^2] \\ &+ \tfrac{1}{2}r^2[(\phi_1')^2 + (\phi_2' - \phi_1')^2 + \ldots + (\phi_{n-1}' - \phi_{n-2}')^2 + (\phi_{n-1}')^2] \\ &+ \tfrac{1}{2}[a_1^2(\phi_2 - 2\phi_1)^2 + a_2^2(\phi_3 - 2\phi_2 + \phi_1)^2 + \ldots + a_{n-1}^2(-2\phi_{n-1} + \phi_{n-2})^2] \\ &+ (1/r^2)[(a_1^2 - n + 1)^2 + (a_2 - a_1 - n + 3)^2 + \ldots + (n - 1 - a_{n-1})^2] \\ &+ \tfrac{2}{3}r^2\{[(\tfrac{1}{2}\phi_1 + \theta)^2 - \eta^2]^2 + [(\tfrac{1}{2}(\phi_2 - \phi_1) + \theta)^2 - \eta^2]^2 + \ldots \\ &+ [(-\tfrac{1}{2}\phi_{n-1} + \theta)^2 - \eta^2]\}\big) \end{aligned} \tag{7}$$

where we recognise by inspection the $(\mathrm{U}(1))^{n-1}$ gauge structure.

The method we use to establish the existence of solutions is that due to Tyupkin *et al* [6] (TFS). To begin with, we know that the Euler-Lagrange equations of the system (2) with respect to the variations $\delta\Theta$, $\delta\Phi$ and δA_i are solved [10, 11] by the Euler-Lagrange equations of the energy integral (7) with respect to the variations $\delta\phi_\alpha$, δa_α, $\alpha = 1, \ldots, n-1$. Thus it is sufficient to prove existence for the problem determined by the functional $E(\phi_\alpha, a_\alpha)$ given by (7). The TFS method consists of verifying that $E(\phi_\alpha, a_\alpha)$ attains its minimal value for some particular set of functions.

Consider first the finite-energy condition $E(\phi_\alpha, a_\alpha) < \infty$. The convergence of the integral as a whole and the positivity of each term in the integrand ensures that each term must converge separately. In particular, the Higgs potential term must converge. This allows us to infer the asymptotic values $\phi_\alpha(\infty)$, $\alpha = 1, \ldots, n-1$. For given n there are many different allowed sets of asymptotic values. However, to satisfy the technical conditions of the TFS method it is necessary to restrict our attention to those asymptotic values which give rise to *non-zero* values for

$$(\phi_2 - 2\phi_1), (\phi_3 - 2\phi_2 + \phi_1), \ldots, (-2\phi_{n-2} + \phi_{n-1}) \tag{8}$$

whose squares are the coefficients of $a_1^2, \ldots, a_{n-1}^2$ in the gauge–Higgs interaction term in (7). A simple analysis shows that, for each n, there are just two sets of solutions (related by a minus sign). However, these solutions exist only if the hitherto arbitrary constant parameter θ takes on specified values. The solutions fall into two classes depending on whether n is even or odd.

n even. In this case $\theta = 0$. We list below the asymptotic values in units of 2η, i.e. $\hat{\phi}_\alpha = \phi_\alpha(\infty)/2\eta$:

$$n=2 \qquad \hat{\phi}_1 = \pm 1$$

$$n=4 \qquad \hat{\phi}_1 = \pm 1 \qquad \hat{\phi}_2 = 0 \qquad \hat{\phi}_3 = \pm 1$$

$$n=6 \qquad \hat{\phi}_1 = \pm 1 \qquad \hat{\phi}_2 = 0 \qquad \hat{\phi}_3 = \pm 1 \qquad \hat{\phi}_4 = 0 \qquad \hat{\phi}_5 = \pm 1.$$

Arbitrary even n:

$$\hat{\phi}_i = \begin{cases} \pm 1 & i \text{ odd} \\ 0 & i \text{ even.} \end{cases} \tag{9}$$

n odd. In this case $\theta = \pm(1/n)\eta$. We list below the asymptotic values $\phi_\alpha(\infty)$ in units of $(4/n)\eta$, i.e. $\hat{\phi}_\alpha = \phi_\alpha(\infty)/(4/n)\eta$:

$$n=3 \qquad \hat{\phi}_1 = \pm 1 \qquad \hat{\phi}_2 = \mp 1$$

$$n=5 \qquad \hat{\phi}_1 = \pm 2 \qquad \hat{\phi}_2 = \mp 1 \qquad \hat{\phi}_3 = \pm 1 \qquad \hat{\phi}_4 = \mp 2$$

$$n=7 \qquad \hat{\phi}_1 = \pm 3 \qquad \hat{\phi}_2 = \mp 1 \qquad \hat{\phi}_3 = \pm 2 \qquad \hat{\phi}_4 = \mp 2$$

$$\hat{\phi}_5 = \pm 1 \qquad \hat{\phi}_6 = \mp 3.$$

Arbitrary odd n:

$$\hat{\phi}_i = \begin{cases} \pm\frac{1}{2}(n-i) & i \text{ odd} \\ \mp\frac{1}{2}i & i \text{ even.} \end{cases} \tag{10}$$

The asymptotic values of the functions $\phi_\alpha(r)$ correspond to the following asymptotic structure for the adjoint Higgs field:

$$n \text{ even} \qquad \Phi(\infty) = \pm \operatorname{diag} 2\eta(1, -1, 1, -1, \ldots, 1, -1) \tag{11a}$$

$$n \text{ odd} \qquad \Phi(\infty) = \pm \operatorname{diag} 2\eta(1-1/n, -(1+1/n), 1-1/n, -(1+1/n), \ldots, -(1+1/n), 1-1/n). \tag{11b}$$

The invariance group of $\Phi(\infty)$ can be read off by inspection: for n even it is $\mathrm{SU}(\frac{1}{2}n) \times \mathrm{SU}(\frac{1}{2}n)$ for $n > 2$ and $\mathrm{U}(1) \times \mathrm{U}(1)$ for $n = 2$, while for n odd it is $\mathrm{SU}(\frac{1}{2}(n+1)) \times \mathrm{SU}(\frac{1}{2}(n-1))$ for $n > 3$ and $\mathrm{SU}(2) \times \mathrm{U}(1)$ for $n = 3$. The proof of existence of field configurations with asymptotic boundary conditions given by (9) or (10) is quite technical [6, 8] and will be reported elsewhere.

It is interesting to note that the solution corresponding to the $n = 2$ case in equation (9) is the Higgs vacuum for the 't Hooft–Polyakov [1] monopole, with a fixed value for the coupling constant, while the $n = 3$ case in equations (10) is the non-self-dual monopole orbit found by Burzlaff [4]. Of course, all the solutions presented here are non-self-dual.

At first glance it would appear that the spherical symmetry ansatz (6) is a $2(n-1)$ function ansatz. However, consistent imposition of the constraint (4) reduces the number of independent functions in the ansatz. Consider the Euler–Lagrange equation for the field ϕ, given by ($3c$),

$$D_iD_i\phi+2(\eta^2+\phi^2)\phi=0. \tag{12}$$

The trace of (12) yields the equation of motion for Θ, namely

$$n\Delta\Theta=-2\,\mathrm{Tr}(\Phi^3+3\Phi^2\Theta)-2n\Theta(\eta^2+\Theta^2). \tag{13}$$

Clearly $\Theta=$ constant is a consistent solution of this equation only if

$$\mathrm{Tr}(\Phi^3+3\Phi^2\Theta)+n\Theta(\eta^2+\Theta^2)=0. \tag{14}$$

Equation (14) acts as a constraint on the ansatz functions ϕ_α, a_α for $\alpha=1,\ldots,n-1$, and will clearly reduce the number of independent functions. Indeed (14) reduces the SU(3) solution to a two-function field configuration, i.e. equivalent to an SU(2) solution, as we show below.

It is actually possible to solve the constraint equation (14). Let us consider first the *odd n* cases. The constraint in this case is then

$$\mathrm{Tr}\,\Phi^2(\mathrm{i}\Phi+(3/n)\eta)=-\eta^3(1-1/n^2). \tag{15}$$

The $n=3$ (i.e. SU(3)) constraint is particularly simple, namely

$$(\phi_1-\tfrac{4}{3}\eta)(\phi_2+\tfrac{4}{3}\eta)(\phi_2-\phi_1-\tfrac{4}{3}\eta)=0. \tag{16}$$

Of the three solutions of (16) one solution, $\phi_2-\phi_1=\frac{4}{3}\eta$, is not consistent with the asymptotic conditions (10), but the remaining two solutions

$$\phi_1=\tfrac{4}{3}\eta \tag{17a}$$

$$\phi_2=-\tfrac{4}{3}\eta \tag{17b}$$

are both acceptable. When these constraints are substituted into the equations of motion for the functions $\phi_1(r)$ and $\phi_2(r)$ they lead, respectively, to the further constraints

$$a_1=0 \tag{18a}$$

$$a_2=0. \tag{18b}$$

Thus, for SU(3), the finite-energy field configuration is parametrised in terms of only two radial functions, either (ϕ_1, a_1) or (ϕ_2, a_2), and not four.

Now let us consider the *even n* cases. The constraint is

$$\mathrm{Tr}\,\Phi^3=0 \tag{19}$$

as, for even n, $\Theta=0$. The $n=2$ case is trivial, as (19) is identically satisfied. The $n=4$ (i.e. SU(4)) constraint is also particularly simple, namely

$$\phi_2(\phi_3-\phi_1)(\phi_1+\phi_3-\phi_2)=0. \tag{20}$$

The solution $\phi_1+\phi_3=\phi_2$ is not consistent with the asymptotic conditions (9) but the remaining two solutions

$$\phi_2=0 \tag{21a}$$

$$\phi_1=\phi_3 \tag{21b}$$

are both acceptable. These solutions lead in turn to the further constraints

$$a_2 = 0 \tag{22a}$$

$$a_1 = \pm a_3 \tag{22b}$$

respectively. Consequently we see that there are two classes of solutions. Class (i) is characterised by $\phi_2 = a_2 = 0$, so that there are four radial functions $(\phi_1, \phi_3, a_1, a_3)$. However, in this case the equations decouple into two identical sets of SU(2) equations for (ϕ_1, a_1) and (ϕ_3, a_3) yielding a two-function, or SU(2), solution. Class (ii), on the other hand, is characterised by four functions $(\phi_1, \phi_2, a_1, a_2)$ for which the equations of motion do not decouple. The class (ii) solutions, then, are true four-function solutions.

For $n > 4$ the constraint equations (15) and (19) can be solved by inspection. For n odd, equation (15) is solved by the following two sets of solutions:

$$\phi_1 = \tfrac{1}{2}(n-1)(4/n)\eta \qquad \phi_3 = \tfrac{1}{2}(n-3)(4/n)\eta \quad \ldots \quad \phi_{n-2} = (4/n)\eta \tag{23a}$$

$$\phi_2 = -(4/n)\eta \qquad \phi_4 = -2(4/n)\eta \quad \ldots \quad \phi_{n-1} = -\tfrac{1}{2}(n-1)(4/n)\eta. \tag{23b}$$

Substitution of (23a) and (23b) in turn into the Euler–Lagrange equations leads to the further constraints

$$a_1 = a_3 = \ldots = a_{n-2} = 0 \tag{24a}$$

$$a_2 = a_4 = \ldots = a_{n-1} = 0. \tag{24b}$$

The resulting equations of motion for the remaining $(n-1)$ functions (ϕ_α, a_α) $\alpha = 2, 4, \ldots, n-1$ or $\alpha = 1, 3, \ldots, n-2$ decouple for each α separately. The pairs of uncoupled equations are all different, however, so that the corresponding field configuration is parametrised by $(n-1)$ different radial functions.

On the other hand, for even n, (19) is solved by each of

$$\phi_2 = \phi_4 = \ldots = \phi_{n-2} = 0 \tag{25a}$$

$$\phi_1 = \phi_3 = \ldots = \phi_{n-3},\ \phi_{n-2} = 0. \tag{25b}$$

When substituted into the equations of motion these constraints lead to

$$a_2 = a_4 = \ldots = a_{n-2} = 0 \tag{26a}$$

$$a_1^2 = a_2^2 = \ldots = a_{n-1}^2 \tag{26b}$$

respectively. When the constraints (25a) and (26a) are combined the equations of motion decouple; the remaining n functions (ϕ_α, a_α) for $\alpha = 1, 3, \ldots, n-1$ satisfy identical differential equations for each value of α. Thus the solutions of the differential equations are identical and we have a two-function field configuration, as seen above for $n = 4$. However, the constraints (25b) and (26b) lead to n differential equations for the $(n-2)$ radial functions (ϕ_α, a_α) $\alpha = 1, 2, \ldots, n-4$ and the pair (ϕ_{n-1}, a_{n-2}) which do not decouple. The resulting field configuration is parametrised by n different radial functions.

We have not found any field configurations parametrised by more than $(n-1)$ different radial functions for n odd, or more than n different radial functions for n even. An exhaustive survey of all the solutions lies beyond the scope of this letter. What we have established, however, is the existence of non-trivial multi-function finite-energy YMH field configurations.

The analysis of this letter can be extended in a number of directions. It is not critical to the TFS method that Θ should be constant. Indeed relaxation of the constraint (4) simplifies the existence proof somewhat. The appropriate asymptotic limits on $\theta(r)$ will be

$$\theta(\infty) = \pm(1/n)\eta \qquad \text{for } n \text{ odd} \tag{27a}$$

$$\theta(\infty) = 0 \qquad \text{for } n \text{ even.} \tag{27b}$$

As the constraints discussed above are not relevant, our method will establish the existence of solutions for a $(2n-1)$-function spherically symmetric ansatz, the $(2n-1)$ functions being $\theta(r)$, $\phi_1(r) \ldots \phi_{n-1}(r)$, $a_1(r) \ldots a_{n-1}(r)$. In this model the Higgs field belongs to the adjoint $\oplus$ scalar representation of $SU(n)$.

The extension of the analysis of this letter to spherically symmetric ansätze different from those of (5) and (6) is being investigated at present and will be reported in a future publication.

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